Linear Nonstationary Models
- A Review of the Work of Professor Phillips -

Katsuto Tanaka
Hitotsubashi University, Tokyo, Japan

Abstract

The work of Professor Phillips, even if it is focused on the area of linear nonstationary models, is enormous. So it is hard for me to explore the whole of his work in this talk. I take up only a few results of his work, which I discuss and, hopefully extend.

1. Topics to be discussed in this talk

[1] **Martingale approximation** (B-N decomposition)

- Time domain and frequency domain:

(1) "Weak Convergence of Sample Covariance Matrices to Stochastic Integrals via Martingale Approximations." Econometric Theory (1988), 4: 528-533

(2) "Asymptotics for Linear Processes." Annals of Statistics (1992), 20: 971-1001 (with Victor Solo)

(3) "Discrete Fourier Transforms of Fractional Processes." CFDP (1999)

2 $K-$**asymptotics**

- Is it possible to differentiate between stochastic and deterministic trends ?

(4) "New Tools for Understanding Spurious Regressions." Econometrica (1998), 66: 1299-1325

(5) "New Unit Root Asymptotics in the Presence of Deterministic Trends," Journal of Econometrics (2002), 111: 323-353

$\boxed{1}$ Martingale approximation

(i) Time domain B-N decomposition

For a $q-$dimensional I(1) process

$$\boldsymbol{y}_j = \boldsymbol{y}_{j-1} + \boldsymbol{u}_j, \quad \boldsymbol{y}_0 = \boldsymbol{0}, \quad (j = 1, \cdots, T),$$

where $\{\boldsymbol{u}_j\}$ is a stationary linear process defined by

$$\boldsymbol{u}_j = \sum_{l=0}^{\infty} A_l \boldsymbol{\varepsilon}_{j-l}, \quad \sum_{l=0}^{\infty} l\,||A_l|| < \infty, \quad A = \sum_{l=0}^{\infty} A_l \neq 0,$$

with $\{\boldsymbol{\varepsilon}_j\} \sim$ i.i.d.$(\boldsymbol{0}, I_q)$, Phillips (1988) proved the following important fact:

$$\frac{1}{T}\sum_{j=1}^{T} \boldsymbol{y}_{j-1}\boldsymbol{u}_j' \;\Rightarrow\; A\int_0^1 \boldsymbol{W}(t)\, d\boldsymbol{W}'(t)\, A' + \Lambda,$$

where $\{\boldsymbol{W}(t)\}$ is the $q-$dimensional standard Brownian motion and

$$\Lambda = \sum_{h=1}^{\infty} \mathsf{E}(\boldsymbol{u}_0\, \boldsymbol{u}_h') = \sum_{h=1}^{\infty} \Gamma(h) = \lim_{T\to\infty} \frac{1}{T} \sum_{j=1}^{T} \mathsf{E}(\boldsymbol{y}_{j-1}\boldsymbol{u}_j').$$

In the proof, the martingale approximation (B-N decomposition) played an important role:

$$\boldsymbol{u}_j = A\,\boldsymbol{\varepsilon}_j + \tilde{\boldsymbol{\varepsilon}}_{j-1} - \tilde{\boldsymbol{\varepsilon}}_j, \qquad \sum_{j=1}^{T} \boldsymbol{u}_j = A \sum_{j=1}^{T} \boldsymbol{\varepsilon}_j + \tilde{\boldsymbol{\varepsilon}}_0 - \tilde{\boldsymbol{\varepsilon}}_T$$

where

$$\tilde{\boldsymbol{\varepsilon}}_j = \sum_{l=0}^{\infty} \tilde{A}_l\, \boldsymbol{\varepsilon}_{j-l}, \quad \tilde{A}_l = \sum_{k=l+1}^{\infty} A_k.$$

For scalar processes, we have

$$\frac{1}{T}\sum_{j=1}^{T} y_{j-1}u_j \;\Rightarrow\; 2\pi f(0)\int_0^1 W(t)\,dW(t) + \lambda,$$

where $f(\omega)$ is the spectrum of $\{u_j\}$ and

$$\lambda = \sum_{h=1}^{\infty} \gamma(h) = \frac{1}{2}\left(2\pi f(0) - \gamma(0)\right).$$

We also have, for $y_j = \rho y_{j-1} + u_j$ with $\rho = 1$,

$$T(\widehat{\rho} - 1) = \frac{\sum_{j=2}^{T} y_{j-1}u_j/T}{\sum_{j=2}^{T} y_{j-1}^2/T^2} \;\Rightarrow\; \frac{\int_0^1 W(t)\,dW(t) + \frac{1}{2}\left(1 - \gamma(0)/(2\pi f(0))\right)}{\int_0^1 W^2(t)\,dt}.$$

As another application, let us consider the unit root seasonal model

$$y_j = \rho_m y_{j-m} + u_j, \quad \rho_m = 1, \quad y_0 = 0, \quad (j = 1, \cdots, T),$$

where m is the period and

$$u_j = \sum_{l=0}^{\infty} \alpha_{lm} \varepsilon_{j-lm}, \quad \sum_{l=1}^{\infty} l\, |\alpha_{lm}| < \infty.$$

Then we have, for $N = [T/m]$,

$$N(\hat{\rho}_m - 1) \Rightarrow \frac{\int_0^1 \boldsymbol{W}'(t)\, d\boldsymbol{W}(t) + \frac{m}{2}\{1 - \gamma(0)/(2\pi f(0))\}}{\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\, dt},$$

where $\{\boldsymbol{W}(t)\}$ is the $m-$dimensional standard Brownian motion.

(ii) Frequency domain B-N decomposition

It is sometimes required to consider the distribution of

$$X_T(\theta) = \sum_{j=1}^{T} u_j \cos j\theta, \qquad Y_T(\theta) = \sum_{j=1}^{T} u_j \sin j\theta, \qquad (0 < \theta < \pi),$$

where $\{u_j\}$ is a stationary linear process defined by

$$u_j = \alpha(L)\,\varepsilon_j = \sum_{l=0}^{\infty} \alpha_l\,\varepsilon_{j-l}, \quad \sum_{l=1}^{\infty} l\,|\alpha_l| < \infty, \quad \sum_{l=0}^{\infty} \alpha_l \neq 0.$$

Asymptotic normality of $X_T(\theta)$ and $Y_T(\theta)$ are proved in T.W. Anderson's 1971 book by a complicated approach. Here we use the martingale approximation for $\{u_j\}$ in the frequency domain.

First of all it holds (Helland 1982, Chan and Wei 1988) that

$$\frac{\sqrt{2}}{\sqrt{T}\sigma}\sum_{j=1}^{[Tt]}\begin{pmatrix}\varepsilon_j \cos j\,\theta \\ \varepsilon_j \sin j\,\theta\end{pmatrix} \Rightarrow \boldsymbol{W}(t), \qquad \frac{\sqrt{2}}{\sqrt{T}\sigma}\sum_{j=1}^{T}\begin{pmatrix}\varepsilon_j \cos j\,\theta \\ \varepsilon_j \sin j\,\theta\end{pmatrix} \Rightarrow \mathrm{N}(\mathbf{0}, I_2),$$

where $\{\boldsymbol{W}(t)\}$ is the two-dimensional standard Brownian motion.

The problem is how to derive the distribution of

$$\frac{\sqrt{2}}{\sqrt{T}\sigma}\sum_{j=1}^{[Tt]}\begin{pmatrix}u_j \cos j\,\theta \\ u_j \sin j\,\theta\end{pmatrix}, \qquad u_j = \sum_{l=0}^{\infty} \alpha_l\,\varepsilon_{j-l}$$

For this purpose the complex B-N decomposition is useful.

$$\boxed{u_j = \alpha(L)\varepsilon_j = \alpha(e^{i\theta})\varepsilon_j + e^{-i\theta}\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_j,}$$

$$\tilde{\varepsilon}_j = \sum_{l=0}^{\infty} \tilde{\alpha}_l \varepsilon_{j-l}, \qquad \tilde{\alpha}_l = \sum_{k=l+1}^{\infty} \alpha_k e^{i(k-l)\theta}.$$

$$\begin{aligned} \sum_{j=1}^{T} e^{ij\theta} u_j &= \alpha(e^{i\theta}) \sum_{j=1}^{T} e^{ij\theta} \varepsilon_j + \sum_{j=1}^{T} e^{i(j-1)\theta} \tilde{\varepsilon}_{j-1} - \sum_{j=1}^{T} e^{ij\theta} \tilde{\varepsilon}_j \\ &= \alpha(e^{i\theta}) \sum_{j=1}^{T} e^{ij\theta} \varepsilon_j + \tilde{\varepsilon}_0 - e^{iT\theta} \tilde{\varepsilon}_T \end{aligned}$$

$$\sum_{j=1}^{T} \begin{pmatrix} u_j \cos j\,\theta \\ u_j \sin j\,\theta \end{pmatrix} = \begin{pmatrix} a(\theta) & -b(\theta) \\ b(\theta) & a(\theta) \end{pmatrix} \sum_{j=1}^{T} \begin{pmatrix} \varepsilon_j \cos j\,\theta \\ \varepsilon_j \sin j\,\theta \end{pmatrix} + \begin{pmatrix} R_{1T} \\ R_{2T} \end{pmatrix}$$

Substituting the complex B-N decomposition, we obtain

$$\frac{\sqrt{2}}{\sqrt{T}\sigma}\sum_{j=1}^{[Tt]}\begin{pmatrix} u_j \cos j\,\theta \\ u_j \sin j\,\theta \end{pmatrix} \quad \Rightarrow \quad K(\theta)\,\boldsymbol{W}(t), \quad (0 \le t \le 1),$$

where

$$K(\theta) = \begin{pmatrix} a(\theta) & -b(\theta) \\ b(\theta) & a(\theta) \end{pmatrix}, \quad a(\theta) = \mathrm{Re}[\alpha(e^{i\theta})], \quad b(\theta) = \mathrm{Im}[\alpha(e^{i\theta})].$$

The above result yields

$$\frac{\sqrt{2}}{\sqrt{T}\sigma}\sum_{j=1}^{T}\begin{pmatrix} u_j \cos j\,\theta \\ u_j \sin j\,\theta \end{pmatrix} \Rightarrow \mathrm{N}(\mathbf{0}, K'(\theta)\,K(\theta)) = \mathrm{N}\left(\mathbf{0}, \frac{2\pi}{\sigma^2} f(\theta)\, I_2\right).$$

More generally, let us consider trigonometric coefficients defined by

$$A(\theta_k) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} u_j \cos j\,\theta_k, \; B(\theta_k) = \frac{1}{\sqrt{T}} \sum_{j=1}^{T} u_j \sin j\,\theta_k, \; (k = 1, \cdots, n),$$

where $0 < \theta_1 < \cdots < \theta_n < \pi$. Then we have

$$A(\theta_1), B(\theta_1), \cdots, A(\theta_n), B(\theta_n) \; \Rightarrow \; \mathsf{N}(\mathbf{0}, D),$$

where

$$D = \mathrm{diag}\,(\pi f(\theta_1), \pi f(\theta_1), \pi f(\theta_n), \pi f(\theta_n))$$

As another application, let us consider the following model

$$(1 - e^{i\theta}L)(1 - e^{-i\theta}L)\, y_j = u_j \quad \Leftrightarrow \quad y_j = \phi_1 y_{j-1} + \phi_2 y_{j-2} + u_j,$$

where $\phi_1 = 2\cos\theta$, $\phi_2 = -1$, and $\{\varepsilon_j\} \sim$ i.i.d.$(0, \sigma^2)$, whereas

$$u_j = \alpha(L)\,\varepsilon_j = \sum_{l=0}^{\infty} \alpha_l \varepsilon_{j-l}, \quad \sum_{l=1}^{\infty} l|\alpha_l| < \infty.$$

The above model is an extension of the model with the i.i.d. error term discussed in Ahtola and Tiao (1987).

Our interest here is to derive the asymptotic distribution of the LSEs of ϕ_1 and ϕ_2.

Assuming that $y_{-1} = y_0 = 0$, we have

$$\begin{aligned} y_j &= \frac{u_j}{(1 - e^{i\theta}L)(1 - e^{-i\theta}L)} = \frac{1}{\sin\theta}\Big[\, z_{1j}\sin(j+1)\theta - z_{2j}\cos(j+1)\theta \,\Big] \\ &= \frac{1}{\sin\theta}\boldsymbol{a}_j'\boldsymbol{z}_j, \end{aligned}$$

where

$$\boldsymbol{z}_j = \begin{pmatrix} z_{1j} \\ z_{2j} \end{pmatrix} = \sum_{l=1}^{j} \begin{pmatrix} \cos l\theta \\ \sin l\theta \end{pmatrix} u_l, \quad \boldsymbol{a}_j = \begin{pmatrix} \sin(j+1)\theta \\ -\cos(j+1)\theta \end{pmatrix}.$$

Applying the complex B-N decomposition to u_l in the expression for $\boldsymbol{z}_j$, we obtain

$$\boldsymbol{z}_j = \sum_{l=1}^{j} \begin{pmatrix} \cos l\theta \\ \sin l\theta \end{pmatrix} u_l = K(\theta)\boldsymbol{x}_j + \boldsymbol{w}_j,$$

where

$$\boldsymbol{x}_j = \sum_{l=1}^{j} \begin{pmatrix} \cos l\theta \\ \sin l\theta \end{pmatrix} \varepsilon_l, \quad \boldsymbol{w}_j = \begin{pmatrix} \mathrm{Re}[\tilde{\varepsilon}_0 - e^{ij\theta}\tilde{\varepsilon}_j] \\ \mathrm{Im}[\tilde{\varepsilon}_0 - e^{ij\theta}\tilde{\varepsilon}_j] \end{pmatrix},$$

$$K(\theta) = \begin{pmatrix} a(\theta) & -b(\theta) \\ b(\theta) & a(\theta) \end{pmatrix}, \quad a(\theta) = \mathrm{Re}[\alpha(e^{i\theta})], \quad b(\theta) = \mathrm{Im}[\alpha(e^{i\theta})].$$

$$
\begin{aligned}
R_T(h) &= \frac{1}{T^2}\sum_{j=h+1}^{T} y_{j-h}\, y_j \;\Rightarrow\; \frac{\pi f(\theta)\cos h\theta}{2\sin^2\theta}\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\,dt,\\
S_T(h) &= \frac{1}{T}\sum_{j=h+1}^{T} y_{j-h}\, u_j\\
&\Rightarrow \frac{1}{\sin\theta}\left[\pi f(\theta)\int_0^1 \boldsymbol{W}'(t)\, J_h(\theta)\, d\boldsymbol{W}(t) + \sum_{j=h}^{\infty}\gamma(j)\sin(j-h+1)\theta\right],
\end{aligned}
$$

where

$$
J_h(\theta) = \begin{pmatrix} -\sin(h-1)\theta & \cos(h-1)\theta \\ -\cos(h-1)\theta & -\sin(h-1)\theta \end{pmatrix}.
$$

Denoting by $\widehat{\boldsymbol{\phi}}$ the LSE of $\boldsymbol{\phi}$ in the present model, we can now establish that

$$
\begin{aligned}
T(\widehat{\boldsymbol{\phi}}-\boldsymbol{\phi}) &= \left[\frac{1}{T^2\sigma^2}\sum_{j=3}^{T}\begin{pmatrix} y_{j-1}^2 & y_{j-1}y_{j-2} \\ y_{j-1}y_{j-2} & y_{j-2}^2 \end{pmatrix}\right]^{-1}\left[\frac{1}{T\sigma^2}\sum_{j=3}^{T}\begin{pmatrix} y_{j-1}u_j \\ y_{j-2}u_j \end{pmatrix}\right] \\
&\Rightarrow \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
Z_1 &= 2\left[\pi f(\theta)\int_0^1 \boldsymbol{W}'(t)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} d\boldsymbol{W}(t)\right. \\
&\qquad \left. +\sin\theta\sum_{j=1}^{\infty}\gamma(j)\cos(j-1)\theta\right]\Bigg/\left\{\pi f(\theta)\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\,dt\right\}, \\
Z_2 &= \frac{-2\left[\int_0^1 \boldsymbol{W}'(t)\,d\boldsymbol{W}(t)+1-\gamma(0)/\left(2\pi f(\theta)\right)\right]}{\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\,dt}.
\end{aligned}
$$

- $(1-e^{i\theta}L)(1-e^{-i\theta}L)\,y_j = \varepsilon_j \quad \Leftrightarrow \quad y_j = \phi_1 y_{j-1} + \phi_2 y_{j-2} + \varepsilon_j,$

$$T(\hat{\phi}_2 + 1) \;\Rightarrow\; \frac{-2\int_0^1 \boldsymbol{W}'(t)\, d\boldsymbol{W}(t)}{\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\, dt}$$

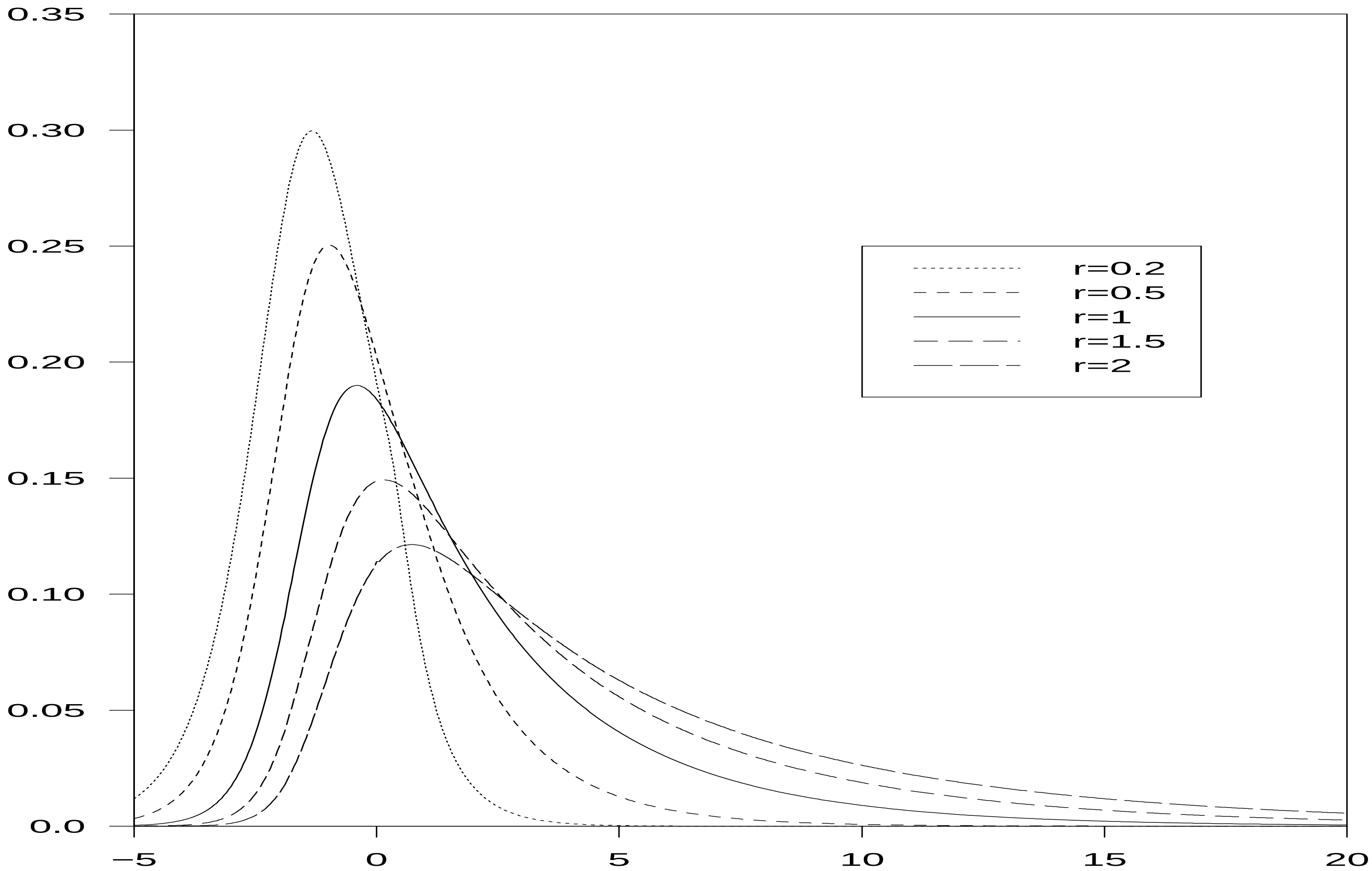
- $(1-e^{i\theta}L)(1-e^{-i\theta}L)\,y_j = u_j \quad \Leftrightarrow \quad y_j = \phi_1 y_{j-1} + \phi_2 y_{j-2} + u_j,$

$$T(\hat{\phi}_2 + 1) \;\Rightarrow\; \frac{-2\left[\int_0^1 \boldsymbol{W}'(t)\, d\boldsymbol{W}(t) + 1 - \gamma(0)/\left(2\pi f(\theta)\right)\right]}{\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\, dt}$$

- $y_j = \rho_m\, y_{j-m} + u_j, \quad \rho_m = 1, \quad y_0 = 0, \quad (j = 1, \cdots, T),$

$$N(\hat{\rho}_m - 1) \;\Rightarrow\; \frac{\int_0^1 \boldsymbol{W}'(t)\, d\boldsymbol{W}(t) + \frac{m}{2}\{1 - \gamma(0)/(2\pi f(0))\}}{\int_0^1 \boldsymbol{W}'(t)\boldsymbol{W}(t)\, dt}$$

Complex unit root distribution: $T(\hat{\phi}_2 + 1)$



$\boxed{2}$ $K-$asymptotics

Let us consider an I(1) process

$$y_j = y_{j-1} + u_j, \qquad y_0 = 0, \qquad (j = 1, \ldots, T),$$

where $\{u_j\}$ is a stationary linear process defined by

$$u_j = \sum_{l=0}^{\infty} \alpha_l \varepsilon_{j-l}, \quad \sum_{l=1}^{\infty} l\,|\alpha_l| < \infty, \quad \alpha \equiv \sum_{j=0}^{\infty} \alpha_j \neq 0,$$

with $\{\varepsilon_j\} \sim$ i.i.d.$(0,\, \sigma^2)$. We also define the partial sum process

$$X_T(t) = \frac{1}{\sqrt{T}\sigma_L} y_{[Tt]} = \frac{1}{\sqrt{T}\sigma_L} \sum_{j=1}^{[Tt]} u_j, \quad (0 \le t \le 1),$$

where $\sigma_L^2 = \alpha^2 \sigma^2$ is the long-run variance of $\{u_j\}$. Then the following FCLT holds:

$$X_T(\cdot) \Rightarrow W(\cdot), \quad \{W(t)\} : \text{standard Brownian motion on } [0,1].$$

This is a typical invariance principle in the weak version, while the strong version [Csörgő and Horváth (1993)] says that, if $\mathrm{E}(|\varepsilon_j|^p) < \infty$ for some $p > 2$, we can construct a standard Brownian motion such that

$$\sup_{0\le t\le 1} T^\delta \, |X_T(t) - W(t)| = \sup_{0\le t\le 1} T^\delta \left| \frac{1}{\sqrt{T}\sigma_L} y_{[Tt]} - W(t) \right| \longrightarrow 0$$

with probability 1, where $0 < \delta = 1/2 - 1/p < 1/2$.

On the other hand, it is known [Loève (1978), Chan and Wei (1988)] that $W(t)$ admits infinitely many ways of series representations. For example, we have

$$W(t) = \sum_{k=1}^{m} g_k(t)\nu_k + \sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}}\xi_n \,,$$

where $\{\nu_k\} \sim$ NID(0,1), $\{\xi_n\} \sim$ NID(0,1), and the two sequences are independent of each other, whereas $g_k(t)$ is a continuous function. Moreover, λ_n is the $n-$th smallest eigenvalue and $f_n(t)$ is the corresponding orthonormal eigenfunction for the integral equation:

$$f(t) = \lambda \int_0^1 K(s,t)\, f(s)\, ds,$$

where $K(s,t)$ is the positive definite kernel defined by

$$K(s,t) = \text{Cov}\left(W(s) - \sum_{k=1}^{m} g_k(s)\nu_k, W(t) - \sum_{l=1}^{m} g_l(t)\nu_l \right) .$$

$$W(t) = \sum_{k=1}^{m} g_k(t)\nu_k + \sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}}\xi_n \, ,$$

Thus, allowing for various values of m and various functions $g_k(t)$, we obtain infinitely many ways of series representations for $W(t)$ that converge with probability 1 and in mean square, uniformly in $t \in$ [0,1]. Among such representations the most convenient for the present purpose is

$$W(t) = \sum_{n=1}^{\infty} \frac{\phi_n(t)}{(n-1/2)\pi}\xi_n \, , \qquad \phi_n(t) = \sqrt{2}\,\text{sin}[(n-1/2)\pi t],$$

where $(n-1/2)\pi$ is the square root of the n-th smallest eigenvalue of the positive definite kernel $K(s,t) = \text{Cov}(W(s), W(t)) = \text{min}(s,t)$, while $\phi_n(t)$ is the corresponding orthonormal eigenfunction.

On the basis of the above facts, Phillips (1998) considers approximating the I(1) process, that is, the process that contains purely stochastic trends, by trigonometric functions with stochastic coefficients. More specifically, the following regression was considered:

$$y_j = \sum_{k=1}^{K} \widehat{b}_k \phi_k \left(\frac{j}{T}\right) + \widehat{u}_j,$$

where $\widehat{b}_1, \cdots, \widehat{b}_K$ are LSEs and $\widehat{u}_j$ is the OLS residual.

Under the above setting, Phillips (1998) obtained the following results on T-asymptotics.

(a) $\widehat{b}_1/\sqrt{T}, \cdots, \widehat{b}_K/\sqrt{T}$ tend to be independently distributed as normal.

(b) $\sum_{j=1}^{T} \widehat{u}_j^2 = O_p(T^2)$

(c) $t_{\widehat{b}_k} = O_p(\sqrt{T})$

(d) $R^2 = O_p(1)$

(e) $DW \to 0$ in probability.

We move on to K-asymptotics by letting $T \to \infty$ and then $K \to \infty$. It holds that

(a) $\widehat{b}_1/\sqrt{T}, \cdots, \widehat{b}_K/\sqrt{T}$ tend to be independently distributed as normal.

(b) $\sum_{j=1}^{T} \widehat{u}_j^2/T^2 = O_p(1/K)$,

(c) $t_{\widehat{b}_k}/\sqrt{T} = O_p(\sqrt{K})$,

(d) $R^2 \to 1$ in probability,

(e) $T \times DW = O_p(K)$.

All of the above statistics signal that the regression relation is valid in K-asymptotics.

It is of great interest to study K-asymptotics in nested models for unit root tests. To this end we consider the regression relation

$$y_j = \widehat{\rho} y_{j-1} + \sum_{k=1}^{K} \widehat{b}_k \phi_k \left(\frac{j}{T} \right) + \widehat{u}_j$$

We first deal with T-asymptotics, for which Phillips (2002) proved that it holds that, as $T \to \infty$,

$$ADF_\rho,\, Z_\rho \quad \Rightarrow \quad \frac{\int_0^1 W_{\boldsymbol{\phi}K}(t)\, dW(t)}{\int_0^1 W^2_{\boldsymbol{\phi}K}(t)\, dt}, \tag{1}$$

$$ADF_t,\, Z_t \quad \Rightarrow \quad \frac{\int_0^1 W_{\boldsymbol{\phi}K}(t)\, dW(t)}{\left(\int_0^1 W^2_{\boldsymbol{\phi}K}(t)\, dt \right)^{1/2}}, \tag{2}$$

- $T-$asymptotics for nested models:

1. $\widehat{b}_k = O_p\left(\frac{1}{\sqrt{T}}\right)$ and does not tend to normality,

2. $t_{\widehat{b}_k} = O_p(1)$

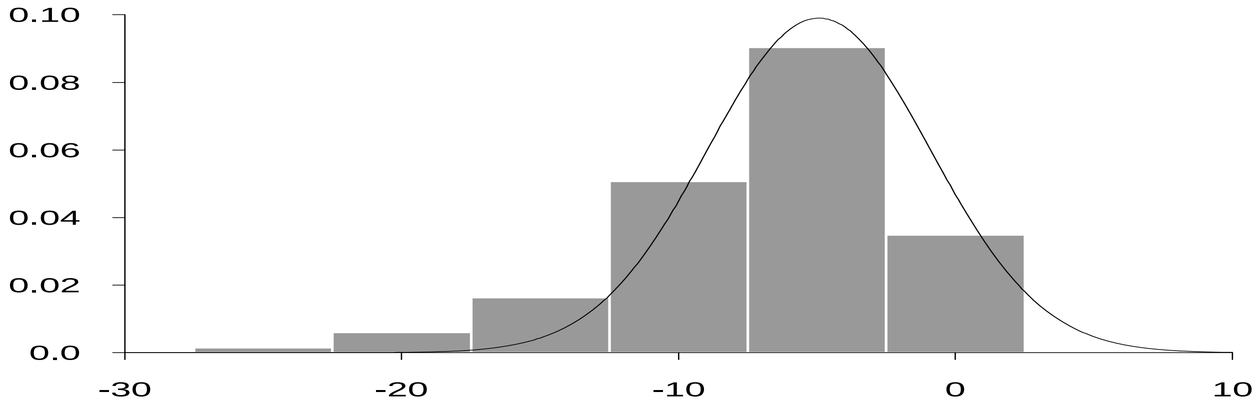
- $K-$asymptotics for nested models:

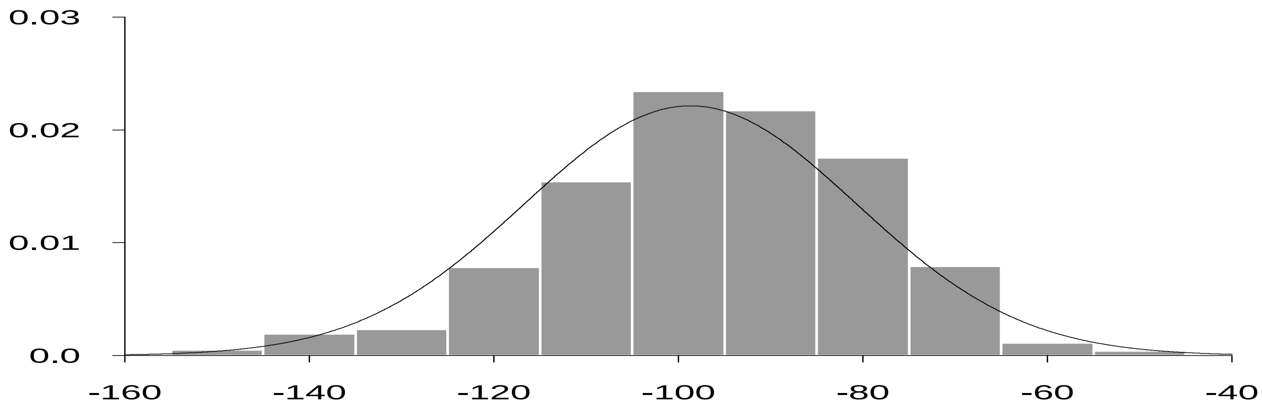
1. $ADF_\rho, Z_\rho \quad \Rightarrow \quad \mathsf{N}\left(-\frac{\pi^2 K}{2}, \frac{\pi^4 K}{6}\right)$,

2. $ADF_t, Z_t \quad \Rightarrow \quad \mathsf{N}\left(-\frac{\pi\sqrt{K}}{2}, \frac{\pi^2}{24}\right)$,

3. $\widehat{b}_k = O_p\left(\frac{1}{\sqrt{T}}\right)$ and tends to normality, $\quad t_{\widehat{b}_k} = O_p(\sqrt{K})$.

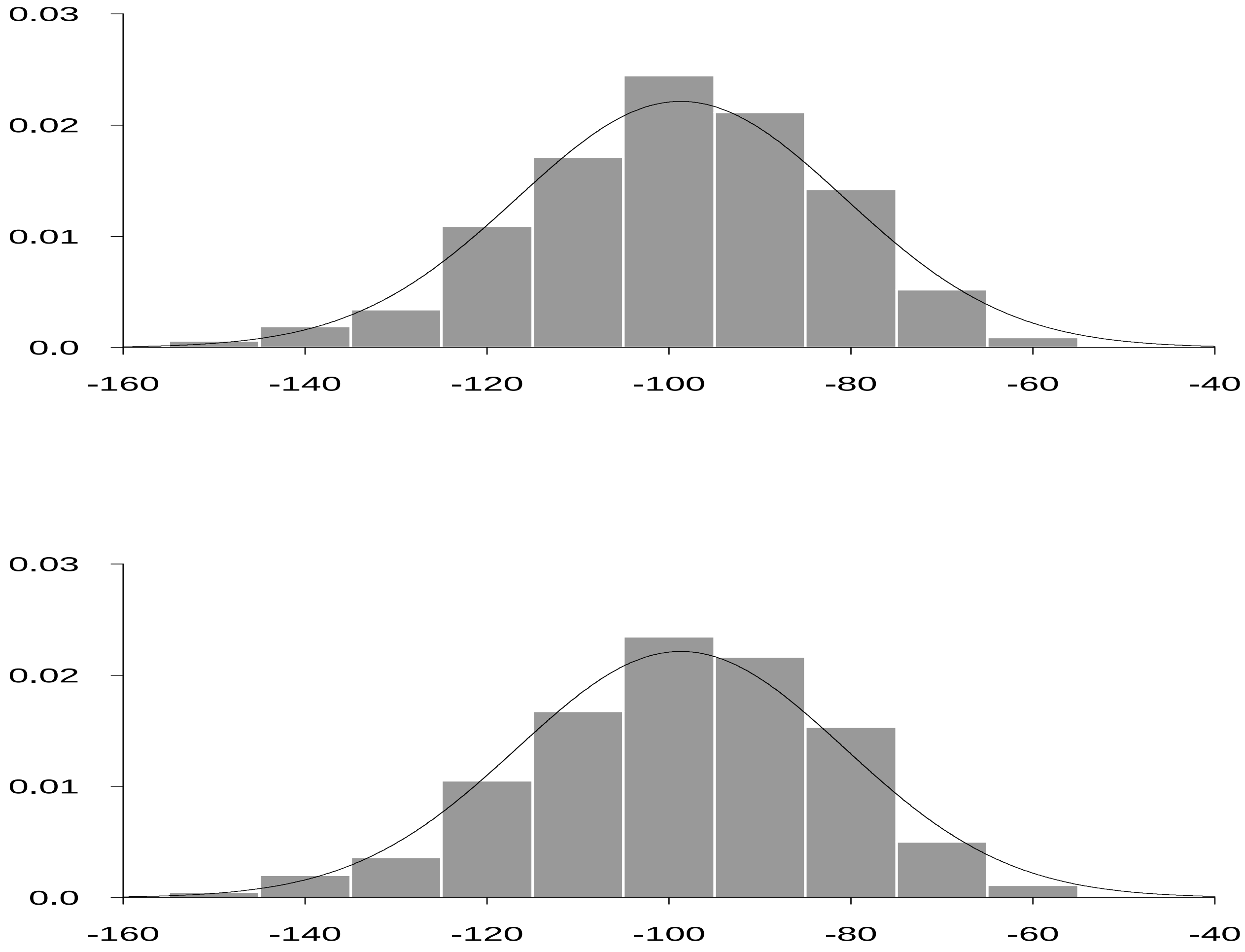
$T(\hat{\rho} - 1) \quad (T = 400, \rho = 1)$ trigonometric trend





top: $K = 1$ bottom: $K = 20$

Power performance ($\rho = 0.95$)



top: true $f_n(t)$ bottom: $f_n(t) = \phi_n(t)$

For a near-integrated process

$$y_j = \left(1 - \frac{c}{T}\right) y_{j-1} + u_j, \quad y_0 = 0,$$

we have

$$Y_T(t) = \frac{1}{\sqrt{T}\sigma_L} \sum_{j=1}^{[Tt]} y_j \Rightarrow Y(t) = e^{-ct} \int_0^t e^{cs}\, dW(s).$$

Here the O-U process $\{Y(t)\}$ admits a series representation:

$$Y(t) = \sum_{n=1}^{\infty} \frac{f_n(t)}{\sqrt{\lambda_n}} \xi_n,$$

where λ_n is the $n-$th smallest eigenvalue of the kernel

$$K(s,t) = \mathrm{Cov}(Y(s), Y(t)) = \frac{e^{-c|s-t|} - e^{-c(s+t)}}{2\,c}.$$

It is found that λ_n is the $n-$th smallest positive solution to

$$\tan \sqrt{\lambda - c^2} = -\frac{\sqrt{\lambda - c^2}}{c}.$$

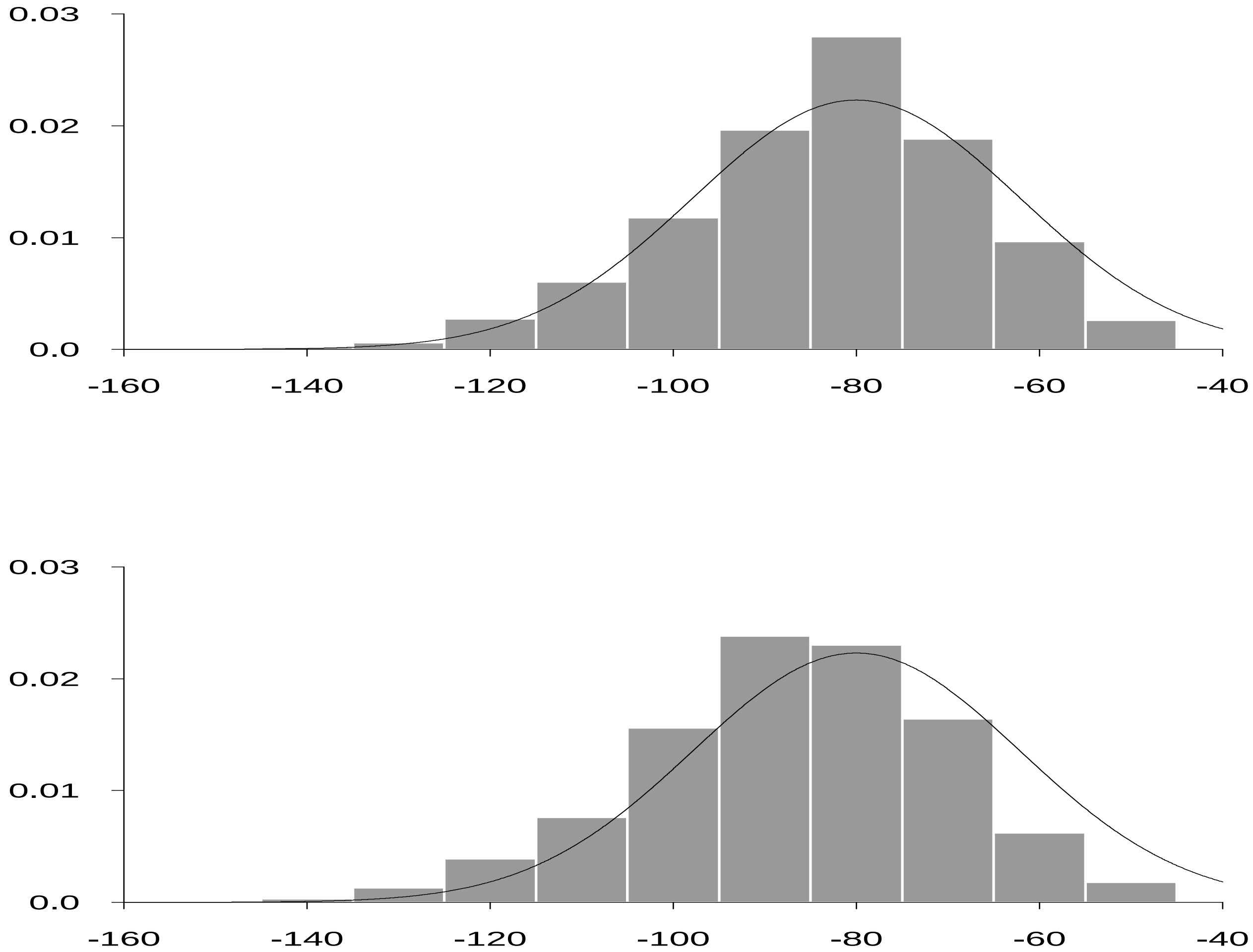
We also obtain

$$f_n(t) = \frac{\sin \mu_n t}{M_n}, \quad \mu_n = \sqrt{\lambda_n - c^2}, \quad M_n = \sqrt{\frac{1}{2} - \frac{\sin 2\mu_n}{4\mu_n}}$$

Then we consider the regression relation

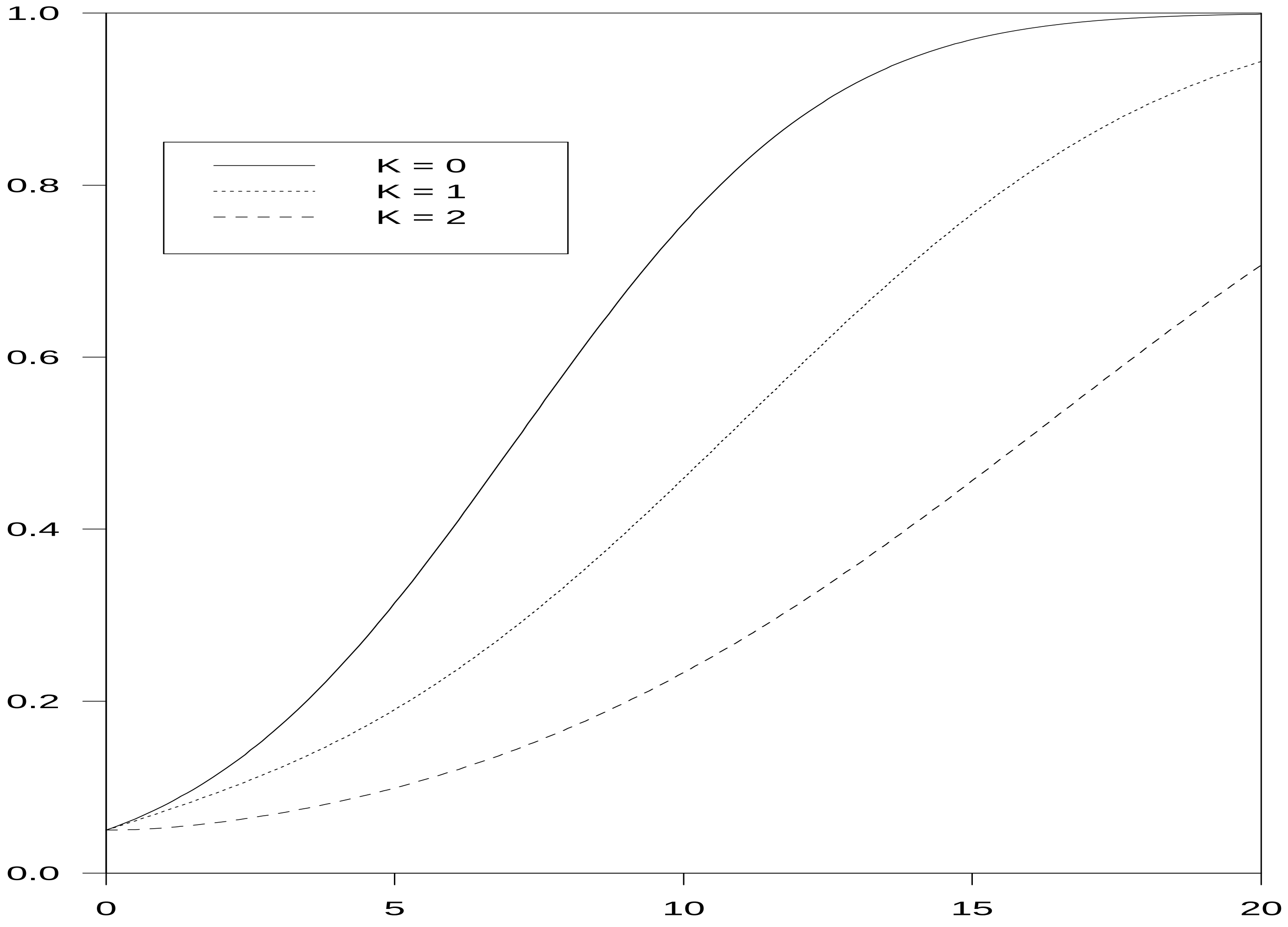
$$y_j = \hat{\rho} y_{j-1} + \sum_{k=1}^{K} \hat{b}_k f_k(j/T) + \hat{u}_j.$$

Power performance (Polynomial trends)



top: $\rho = 1$ bottom: $\rho = 0.95$

Limiting local powers of the D-F test for Models with $K = 0$, 1, and 2



• Conclusions on K-asymptotics:

When the DGP of $\{y_j\}$ is I(1), that is, $y_j = y_{j-1} + u_j$, the following facts are observed:

i) Deterministic trends of K components tend to explain fully the true process $\{y_j\}$.

ii) Deterministic trends are still significant if y_j is regressed on y_{j-1} in addition to deterministic trends of K components.

iii) The unit root test based on the regression of y_j on y_{j-1} and detrerministic trends of K components loses its power against near integration since the unit root null distribution is asymptotically the same as the local alternative distribution.