Peter C.B. Phillips and Nonstationary Nonlinearity

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Contents

References

Phillips and Park (1998)	Nonstationary Density Estimation and Kernel Autoregression Cowles Foundation Working Paper
Park and Phillips (1999)	Asymptotics for Nonlinear Transformations of Integrated Time Series *Econometric Theory*, 15, 269-298
Park and Phillips (2001)	Nonlinear Regressions with Integrated Time Series *Econometrica*, 69, 1452-1498
Park and Phillips (2000)	Nonstationary Binary Choice *Econometrica*, 68, 1249-1280

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Hu and Phillips (2004)	Nonstationary Discrete Choice *Journal of Econometrics,* 120, 103-138
Phillips, Jin and Hu (2007)	Nonstationary Discrete Choice: A Corrigendum and Addendum *Journal of Econometrics,* 141, 1115-1130

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Hong and Phillips (2006)	Testing Linearity in Cointegrating Relations with an Application to PPP Mimeograph

Basic Asymptotics

D G P

Consider a time series generated by

$$x_t = x_{t-1} + v_t \ , \quad v_t \ \ iid\left(0, \sigma^2\right)$$

May allow for more general processes having serial dependence and thick tails

Use the notation

$I(1)$ integrated of order one

$I(0)$ stationary

Basic Tools

- Invariance Principle

$$V_n(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} v_t$$

$V_n \to_d V$, V BM with variance σ^2

- Brownian Local Time

$$L(t,s) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1\{|V(r) - s| < \varepsilon\} dr$$

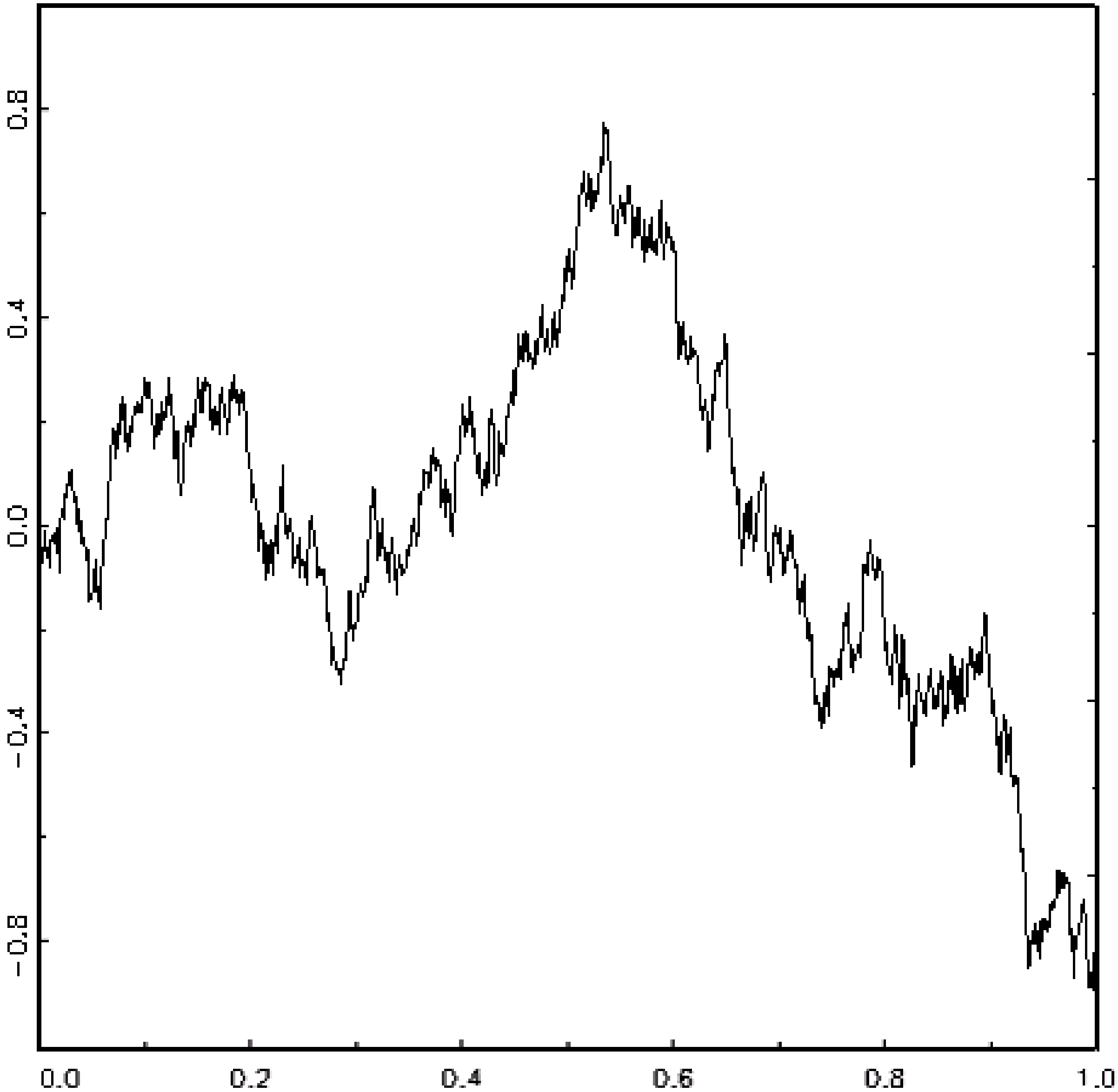
continuous in both t and s

- Occupation Times Formula

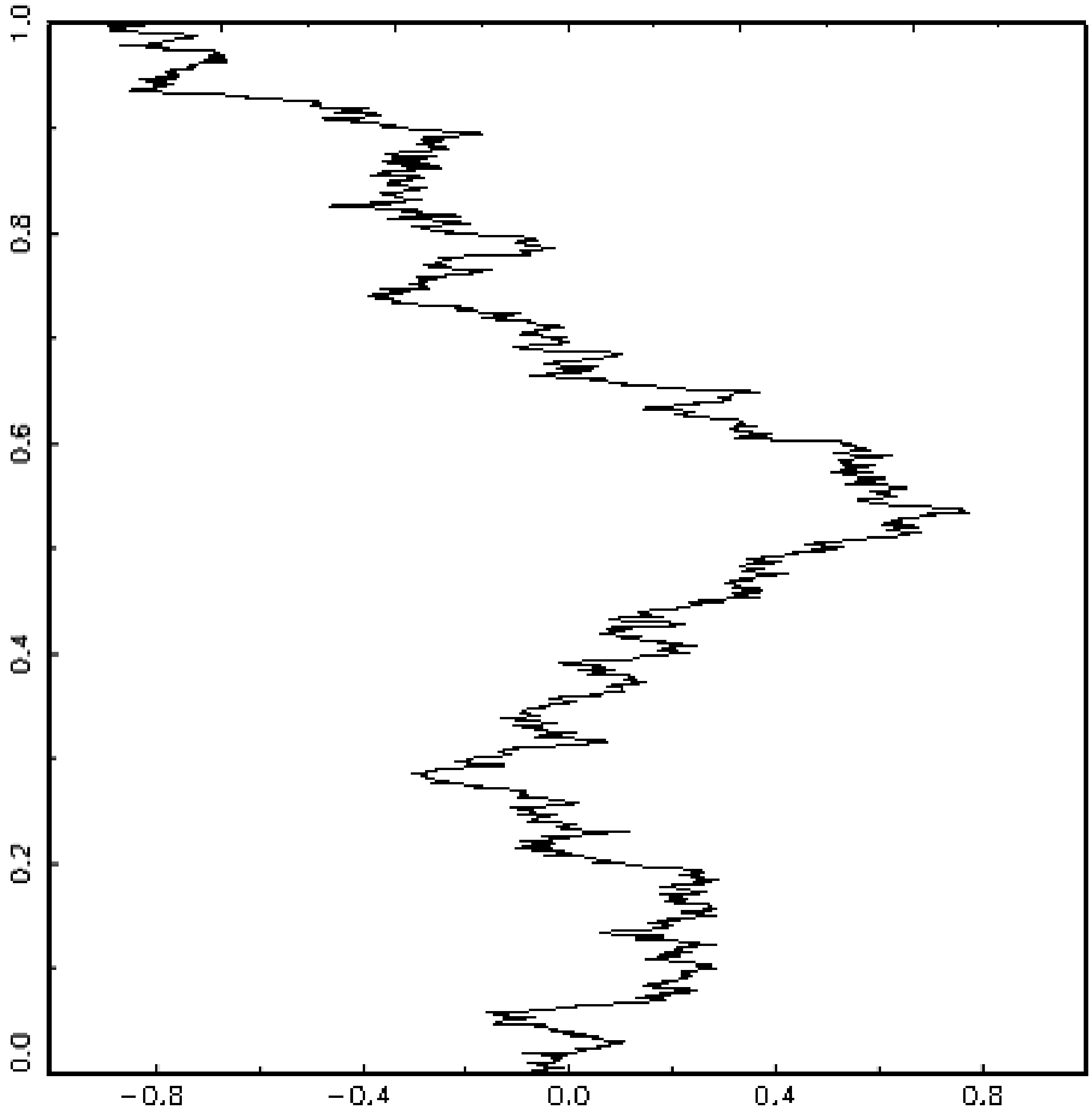
$$\int_0^t F(V(r))\, dr = \int_{-\infty}^{\infty} F(s) L(t,s)\, ds$$

for any locally integrable F

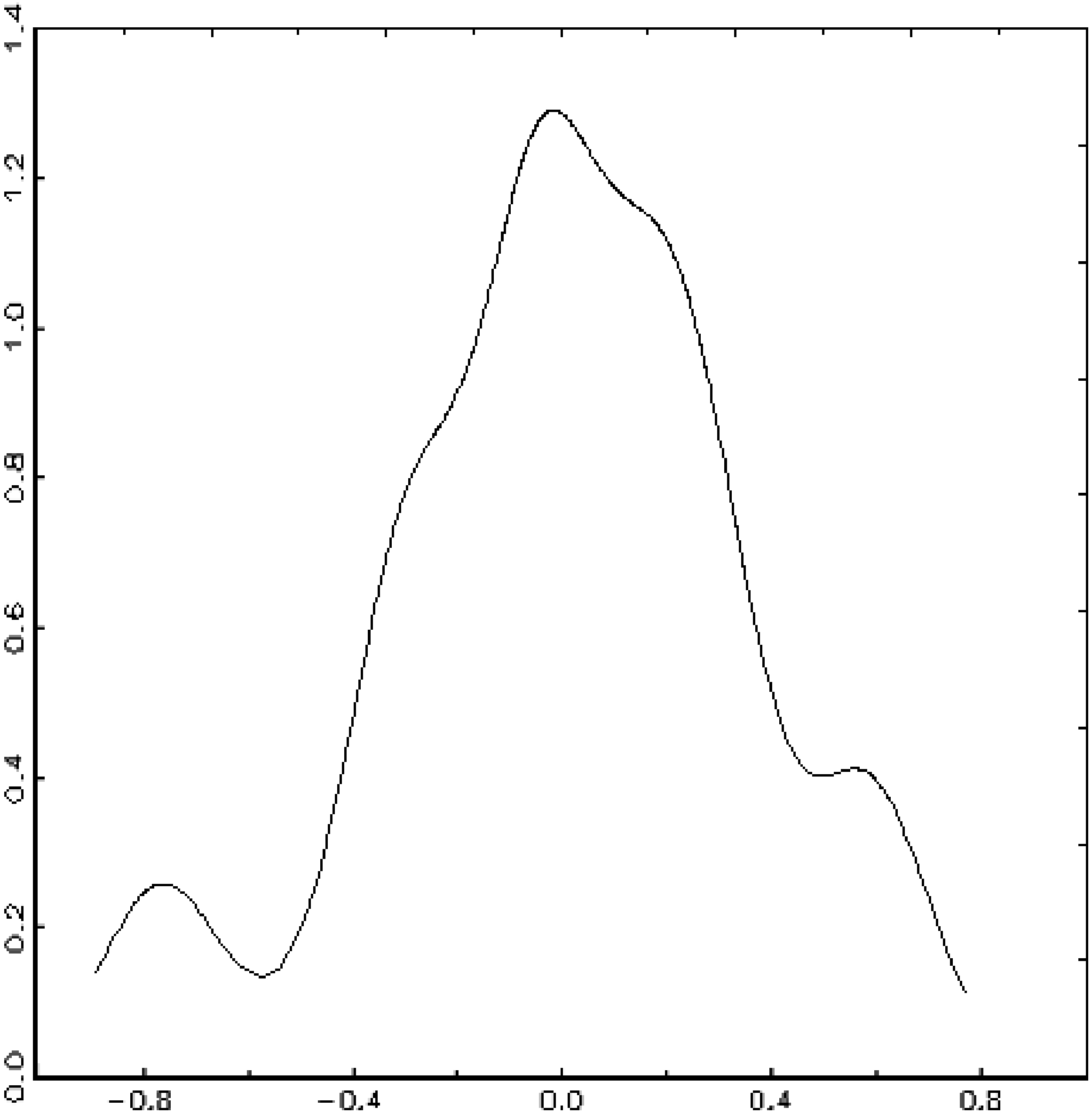
Brownian Sample Path



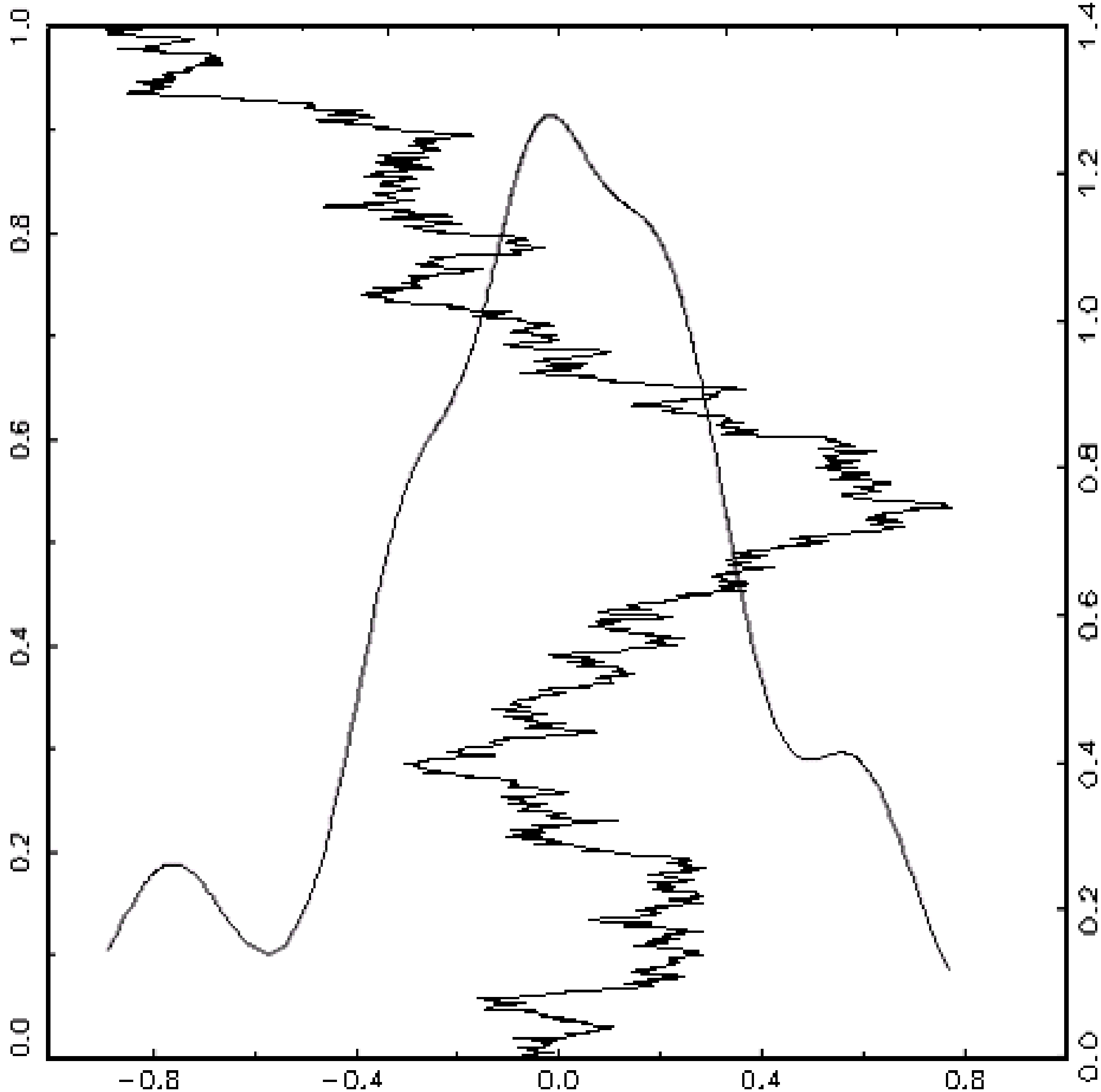
Another Look at Sample Path



Estimated Local Time



Sample Path and Estimated Local Time



More on Local Time

Local Time as Density

$$z_t = V\left(\frac{t}{n}\right), \qquad t = 1, \ldots, n$$

$$\frac{1}{n}\sum_{t=1}^{n} z_t^k \to_{a.s.} \int_0^1 V(r)^k dr = \int_{-\infty}^{\infty} s^k L(1,s)\, ds$$

$$\begin{aligned}
\frac{1}{nh_n}\sum_{t=1}^{n} K\left(\frac{z - z_t}{h_n}\right) &\approx \frac{1}{h_n}\int_0^1 K\left(\frac{z - V(r)}{h_n}\right) dr \\
&= \frac{1}{h_n}\int_{-\infty}^{\infty} K\left(\frac{z - s}{h_n}\right) L(1,s)\, ds \\
&= \int_{-\infty}^{\infty} K(s)\, L(1, z - h_n s)\, ds \\
&\to_{a.s.} L(1,z)
\end{aligned}$$

Function Classes

F Integrable

$$F(x) = 1\{a \le x \le b\}, \quad e^{-x^2}$$

F Asymptotically Homogeneous

$$F(\lambda x) \approx \nu(\lambda) H(x) \quad \text{for } \lambda \text{ large}$$

ν Asymptotic Order

H Limit Homogeneous Function

$$F(x) = |x|^k \qquad |\lambda x|^k = \lambda^k |x|^k$$

$$F(x) = \log|x| \qquad \log|\lambda x| \approx \log\lambda$$

$$F(x) = \frac{e^x}{1+e^x} \qquad \frac{e^{\lambda x}}{1+e^{\lambda x}} \approx 1\{x \ge 0\}$$

Basic Asymptotics

Integrable F

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}F\left(x_t\right)\approx\sqrt{n}\int_0^1 F\left(\sqrt{n}V\left(r\right)\right)dr$$

$$=\sqrt{n}\int_{-\infty}^{\infty}F\left(\sqrt{n}s\right)L\left(1,s\right)ds$$

$$=\int_{-\infty}^{\infty}F\left(s\right)L\left(1,\frac{s}{\sqrt{n}}\right)ds$$

$$\approx L\left(1,0\right)\int_{-\infty}^{\infty}F\left(s\right)ds$$

Asymptotically Homogeneous F

$$\frac{1}{n\,\nu\left(\sqrt{n}\right)}\sum_{t=1}^{n}F\left(x_t\right)\approx\frac{1}{\nu\left(\sqrt{n}\right)}\int_0^1 F\left(\sqrt{n}V\left(r\right)\right)dr$$

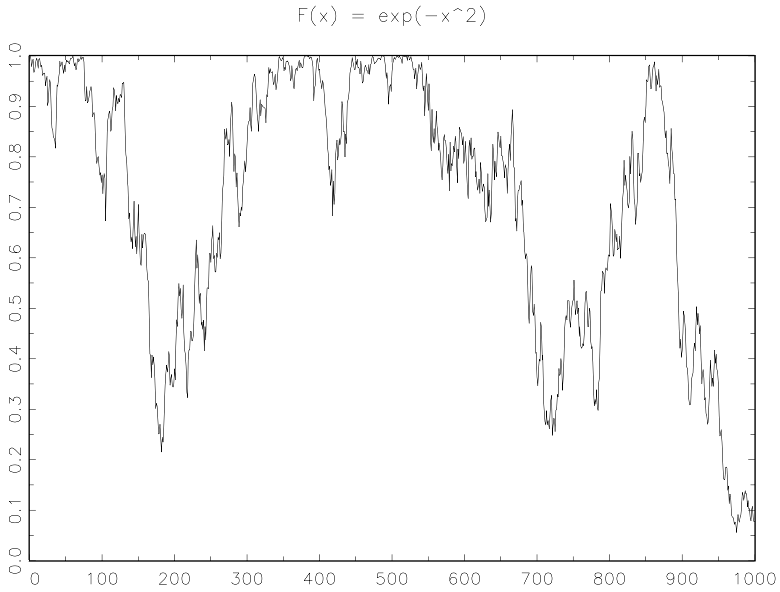
$$\approx\int_0^1 H\left(V\left(r\right)\right)dr$$

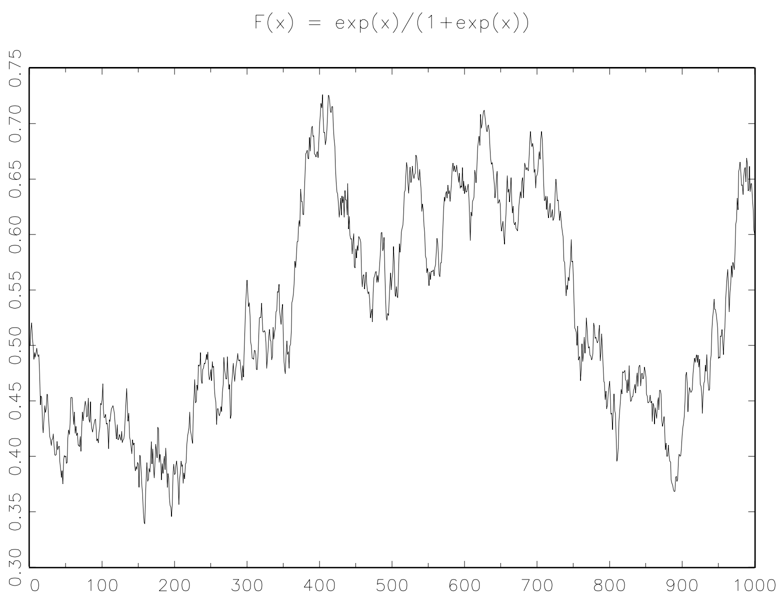
Nonlinear Transformations of Integrated Processes

Data Generation

$$y_t = F(x_t)$$

x_t	$I(1)$	Economic Fundamental
y_t	$I(0)$ or $I(1)$	Observed Time Series
F	Integrable or Asymptotically Homogeneous	Economic Institution and/or Policy Intervention


F(x) = exp(-x^2)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0
100
200
300
400
500
600
700
800
900
1000


F(x) = exp(x)/(1+exp(x))
0.30
0.35
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0
100
200
300
400
500
600
700
800
900
1000

Sample Statistics

- Sample Autocorrelation

$$R_{nk} = \frac{\sum_{t=1}^{n} \left(y_t - \overline{y}_n\right)\left(y_{t-k} - \overline{y}_n\right)}{\sum_{t=1}^{n} \left(y_t - \overline{y}_n\right)^2}$$

- Sample Variance

$$S_n^2 = \frac{1}{n} \sum_{t=1}^{n} \left(y_t - \overline{y}_n\right)^2$$

- Sample Kurtosis

$$K_n^4 = \frac{\frac{1}{n} \sum_{t=1}^{n} \left(y_t - \overline{y}_n\right)^4}{\left(\frac{1}{n} \sum_{t=1}^{n} \left(y_t - \overline{y}_n\right)^2\right)^2}$$

Time Series Properties

- F Integrable

$$R_{nk} \to_p R_k = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x)F(x+y)f_k(y)dxdy}{\int_{-\infty}^{\infty} F(x)^2 dx}$$

f_k density of $x_t - x_{t-k}$

$|R_k| < 1$ for all $k \geq 1$, unit root disappears

$R_k \to 0$ as $k \to \infty$, but very slowly

$R_k \approx k^{-1/2}$ for large k, under Gaussianity

$y_t \approx I(d)$ with $d = 1/4$

stationary long memory generated by nonstationary nonlinearity

$$\sqrt{n}\, S_n^2 \to_d L(1,0)\int_{-\infty}^{\infty} F(x)^2\, dx$$

$$S_n^2 \to_p 0$$

$$\frac{K_n^4}{\sqrt{n}} \to_d \frac{\int_{-\infty}^{\infty} F(x)^4\, dx}{L(1,0)\left(\int_{-\infty}^{\infty} F(x)^2\, dx\right)^2}$$

$$K_n^4 \to_p \infty$$

- F Asymptotically Homogeneous

$$F(\lambda x) \approx \nu(\lambda) H(x) \quad \text{for } \lambda \text{ large}$$

ν Asymptotic Order

H Limit Homogeneous Function

- Case 1 : H constant

$F = c + G$, G Integrable

R_{nk}, S_n^2, and K_n^4 behave as for Integrable F

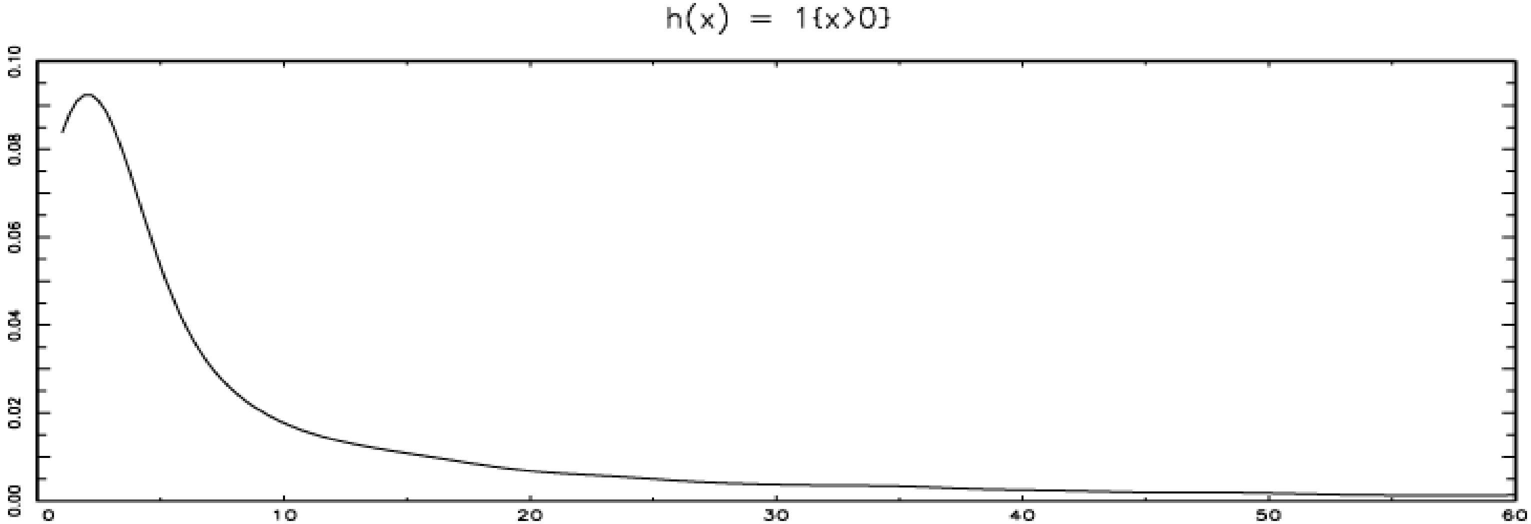
- Case 2 : H nonconstant

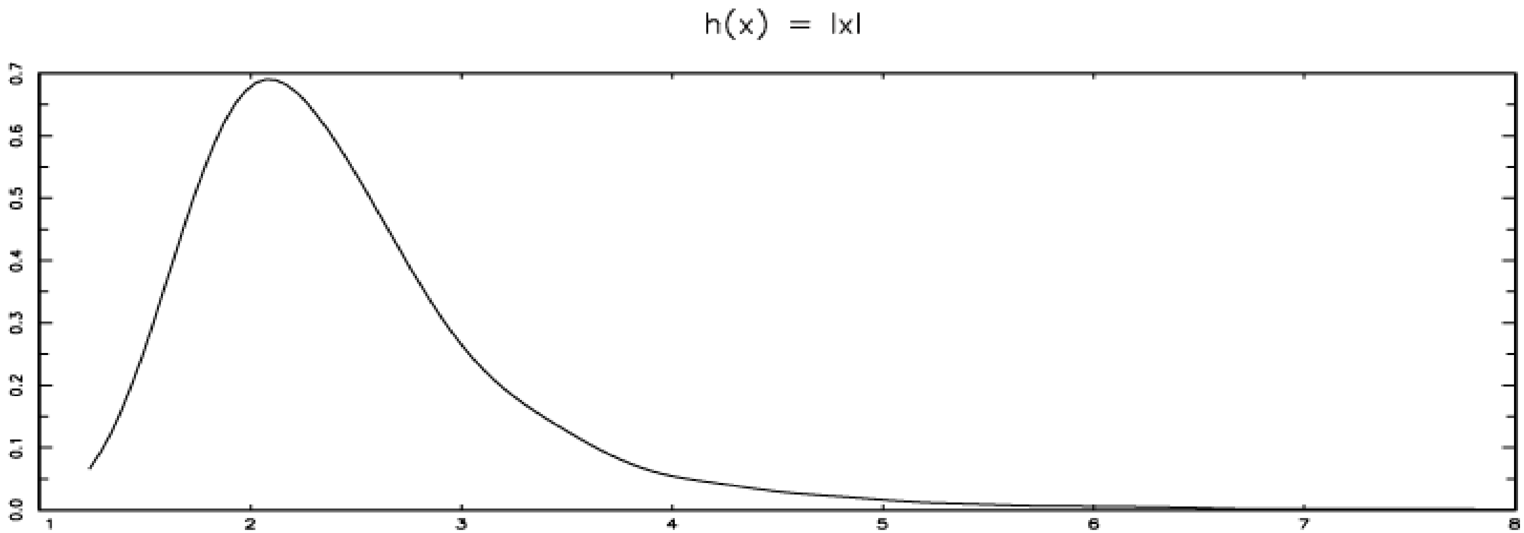
$$R_{nk} \to_p 1 \quad \text{for all } k$$

$$\frac{S_n^2}{\nu\left(\sqrt{n}\right)} \to_d \int_0^1 \left(H(V) - \int_0^1 H(V) \right)^2$$

$$K_n^4 \to_d \frac{\int_0^1 \left(H(V) - \int_0^1 H(V) \right)^4}{\left[\int_0^1 \left(H(V) - \int_0^1 H(V) \right)^2 \right]^2}$$

Asymptotic Densities for Sample Kurtoses





Nonlinear Regressions for Integrated Regressors

Models

$$y_t = F(x_t) + u_t$$ Nonparametric Regression

$$= G(x_t, \theta_0) + u_t$$ Nonlinear Regression

F, G Function Classes	Integrable	Asymptotically Homogeneous
x_t	$I(1)$, F_t -adapted	$I(1)$
y_t	$I(0)$	$I(1)$
u_t	MDS w.r.t. F_t	$I(0)$
Model Interpretation	Cond Mean Model $\mathbf{E}(y_t \mid F_t) = F(x_t)$ $G(x_t, \theta_0)$	Nonlinear or Nonparametric Cointegration

Nonlinear Regression

Asymptotic Equivalence

NLS Regression $\quad y_t = G(x_t, \theta) + u_t$

LS Regression $\quad y_t = \dot{G}(x_t, \theta_0)'\theta + u_t\ , \quad \dot{G} = \frac{\partial}{\partial \theta} G$

NLS Estimator

$$\hat{\theta} \approx \left(\sum_{t=1}^{n} \dot{G}(x_t, \theta_0) \dot{G}(x_t, \theta_0)' \right)^{-1} \sum_{t=1}^{n} \dot{G}(x_t, \theta_0) y_t$$

$$= \theta_0 + \left(\sum_{t=1}^{n} \dot{G}(x_t, \theta_0) \dot{G}(x_t, \theta_0)' \right)^{-1} \sum_{t=1}^{n} \dot{G}(x_t, \theta_0) u_t$$

Main Results

Integrable $\dot{G}$

$$\sqrt[4]{n}\left(\hat{\theta}_n - \theta_0\right) \to_d \left(\sigma^2 L(1,0) \int_{-\infty}^{\infty} \dot{G}(x_t, \theta_0) \dot{G}(x_t, \theta_0)' \, dx\right)^{-1/2} N(0,1)$$

reduced convergence rate

mixed normality

Asymptotically Homogeneous $\dot{G}$

$$\dot{G}(\lambda x, \theta_0) \approx \dot{\nu}(\lambda, \theta_0) \dot{H}(x, \theta_0)$$

$$\sqrt{n}\, \dot{\nu}\left(\sqrt{n}, \theta_0\right)\left(\hat{\theta}_n - \theta_0\right) \to_d \left(\int_0^1 \dot{H}(V) \dot{H}(V)'\right)^{-1} \int_0^1 \dot{H}(V) \, dU$$

convergence rate given by $\dot{\nu}$

nonnormal unless x_t strictly exogenous

Nonparametric Regression

Nadaraya-Watson Estimator

$$\hat{F}_n(x) = \frac{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right) y_t}{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right)}$$

K kernel function, $\int_{-\infty}^{\infty} K(x)dx = 1$

h_n bandwidth, $h_n \to 0 \text{ as } n \to \infty$

$$= \frac{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right) F(x_t)}{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right)} + \frac{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right) u_t}{\sum_{t=1}^{n} K\left(\frac{x - x_t}{h_n}\right)}$$

trend / source of bias (first term)

martingale / source of variance (second term)

Main Results

Trend Part

$$\frac{\sum_{t=1}^{n} K\left(\frac{x-x_t}{h_n}\right) F(x_t)}{\sum_{t=1}^{n} K\left(\frac{x-x_t}{h_n}\right)} \approx F(x) + \frac{1}{2} h_n^2 F''(x) \int_{-\infty}^{\infty} s^2 K(s)\, ds$$

Martingale Part

$$\frac{\sum_{t=1}^{n} K\left(\frac{x-x_t}{h_n}\right) u_t}{\sum_{t=1}^{n} K\left(\frac{x-x_t}{h_n}\right)} \approx n^{-1/4} h_n^{-1/2} \left(\sigma^2 L(1,0)^{-1} \int_{-\infty}^{\infty} K(s)^2\, ds \right)^{1/2} N(0,1)$$

Optimal Bandwidth $\quad h_n = n^{-1/10} \; : \; h_n^2 = n^{-1/4} h_n^{-1/2}$

Rates of Convergence $\quad n^{1/5} \sim n^{1/4}$

Parametric vs Nonparametric Regression

Rates of Convergence

		Parametric	Nonparametric
Stationary Regression		$n^{1/2}$	$n^{2/5} \sim n^{1/2}$
Nonstationary Regression	Asymptotically Homogeneous F	$n^{1/2}\nu\left(n^{1/2}\right)$	$n^{1/5} \sim n^{1/4}$
	Integrable F	$n^{1/4}$	

Nonstationary Nonlinearity as Inferential Instrument

Two Important Characteristics

F Integrable

Asymptotic Mixed Normality

$$\frac{1}{\sqrt[4]{n}}\sum_{t=1}^{n} F\left(x_{t-1}\right)v_t \to_d \left(\sigma^2 L\left(0,1\right)\int_{-\infty}^{\infty} F\left(x\right)^2 dx\right)^{1/2} N\left(0,1\right)$$

asymptotically mixed normal

Asymptotic Orthogonality

$$\frac{1}{\sqrt[4]{n}}\sum_{t=1}^{n} F\left(x_{1,t-1}\right)v_{1t}\ ,\quad \frac{1}{\sqrt[4]{n}}\sum_{t=1}^{n} F\left(x_{2,t-1}\right)v_{2t}$$

asymptotically uncorrelated

even when v_{1t} and v_{2t} are correlated

A Closer Look

Integrable F truncates (or downsizes) large values

Asymptotic Mixed Normality

$$\mathrm{var}\left(\frac{1}{\sqrt[4]{n}} \sum_{t=1}^{n} F(x_{t-1}) v_t \right) = \sigma^2 \mathbf{E}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} F(x_{t-1})^2 \right)$$

Stochastic order of $\sum_{t=1}^{n} F(x_{t-1})^2$

$\approx$ # of x_t taking small values $\approx \sqrt{n}$

Asymptotic Orthogonality

$$\operatorname{cov}\left(\frac{1}{\sqrt[4]{n}} \sum_{t=1}^{n} F\left(x_{1,t-1}\right) v_{1t}, \frac{1}{\sqrt[4]{n}} \sum_{t=1}^{n} F\left(x_{2,t-1}\right) v_{2t} \right)$$

$$= \sigma_{12} \mathbf{E}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} F\left(x_{1,t-1}\right) F\left(x_{2,t-1}\right) \right) \to 0$$

Stochastic order of $\sum_{t=1}^{n} F\left(x_{1,t-1}\right) F\left(x_{2,t-1}\right)$

$\approx$ # of both x_{1t} and x_{1t} taking small values

$\approx \log n$

Applications

Testing for a Unit Root

Test of the hypothesis

$H_0: \ \alpha = 1$ in $x_t = \alpha x_{t-1} + v_t$

IV estimation using $F\left(x_{t-1}\right)$ as an instrument

IV t-ratio

$$\tau = \frac{\sum_{t=1}^{n} F\left(x_{t-1}\right) v_t}{\sigma\left(\sum_{t=1}^{n} F\left(x_{t-1}\right)^2\right)^{1/2}} \to_d N\left(0,1\right)$$

Testing for Unit Roots in Dependent Panels

Test of the hypothesis

$H_0: \ \alpha_i = 1$ in $x_{it} = \alpha_i x_{i,t-1} + v_{it}$

Individual IV t-ratio for each i

$$\tau_i = \frac{\sum_{t=1}^{T} F\left(x_{i,t-1}\right) v_{it}}{\sigma_i \left(\sum_{t=1}^{T} F\left(x_{i,t-1}\right)^2 \right)^{1/2}} \approx_d N(0,1)$$

Average IV t-ratios across i

$$\tau = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \tau_i \approx_d N(0,1)$$

as long as T large

Nonstationary Latent Variables Models

Models

$$y_t^* = x_t^{'}\beta_0 - \varepsilon_t, \quad (\varepsilon_t) \sim iid\ F$$

Binary Choice Model

$$y_t = 1\left\{ y_t^* \geq 0 \right\}$$

General Discrete Choice Model

$$\begin{aligned} y_t &= 0,\ y_t^* \in \left(-\infty, \sqrt{n}\mu_0^1\right] \\ &= 1,\ y_t^* \in \left(\sqrt{n}\mu_0^1, \sqrt{n}\mu_0^2\right] \\ &\vdots \\ &= J,\ y_t^* \in \left(\sqrt{n}\mu_0^J, \infty\right) \end{aligned}$$

Asymptotics

Binary Choice Model

$$\alpha = \left(\alpha_1, \alpha_2^{'}\right)' = \left(h_1^{'}\beta, (H_2^{'}\beta)'\right)' = H'\beta \qquad \textit{rotation}$$

$$D_n = \operatorname{diag}\left(n^{1/4}, n^{3/4}\right) \qquad \textit{normalization}$$

$$D_n\left(\hat{\alpha}_n - \alpha_0\right) \to_d MN \qquad \textit{mixed normal distribution}$$

$$n^{1/4}\left(\hat{\beta}_n - \beta_0\right) \to_d MN \qquad \textit{singular covariance matrix}$$

Asymptotics

General Discrete Choice Model

$$\mu = (\mu_1, \ldots, \mu_J)'$$

$$n^{3/4}\left((\hat{\mu}_n - \mu_0)', (\hat{\beta}_n - \beta_0)'\right)' \to_d MN$$

$$single\, convergence\, rate$$

$$nonsingular\; covariance\, matrix$$